How would you ***start*** these?

1. $\displaystyle\int \frac{x^2+7}{x^2(3-x)}\,dx$ PARTIAL FRACTION $\dfrac{x^2+7}{x^2(3-x)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{3-x}$

2. $\displaystyle\int \sqrt{x}\ln(x)dx$ BY PARTS $u=\ln(x)$ $dv = x^{1/2}dx$ $du = \frac{1}{x}dx$ $v = \frac{2}{3}x^{3/2}$

3. $\displaystyle\int \frac{1}{(x^2+6x+13)^{3/2}}\,dx$ TRIG SUB $x^2+6x+9-9+13 = (x+3)^2+4$ $\displaystyle\int \frac{1}{((x+3)^2+4)^{3/2}}dx$ $x+3 = 2\tan\theta$

4. $\displaystyle\int \tan^{-1}(x)\,dx$ BY PARTS $u=\tan^{-1}(x)$ $dv=dx$ $du = \frac{1}{x^2+1}dx$ $v=x$

5. $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$ TRIG – ODD COSINE $\int \sin^2(x)\cos^2(x)\cos(x)dx$ $u=\sin(x)$ $\int \sin^2(x)(1-\sin^2(x))\cos(x)dx$

6. $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$ TRIG SUB $x=5\sin\theta$

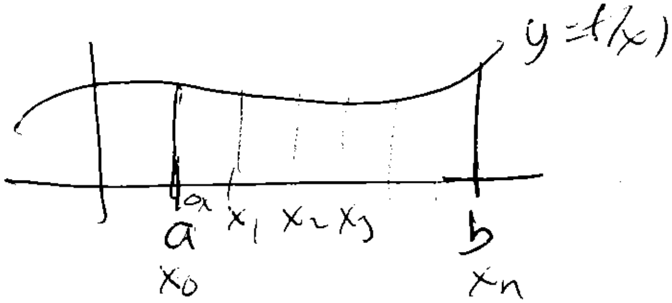
7. $\displaystyle\int \frac{\sqrt{x}}{x-9}dx$ $t=\sqrt{x} \Leftrightarrow t^2=x$ $2t\,dt=dx$ $\displaystyle\int \frac{t}{t^2-9}2t\,dt = \int\frac{2t^2}{t^2-9}dt$ DIVIDE! THEN PARTIAL FRAC

8. $\displaystyle\int \tan^4(x)\sec^4(x)\,dx$ TRIG – EVEN SEC $\int \tan^4(x)\sec^2(x)\sec^2(x)dx$ $u=\tan(x)$ $\int \tan^4(x)(1+\tan^2(x))\sec^2(x)dx$

9. $\displaystyle\int x\sqrt{4-x}\,dx$ $u=4-x$ $du=-dx$ $x=4-u$

7.7 Approximating Integrals

We have learned how to integral some important situations. ***But*** many, many, many integrals CANNOT be done with any of our methods. So, in a great many applications, we have to approximate!



To approximate $\int_a^b f(x)dx$

1. Compute $\Delta x = \frac{b-a}{n}$.

   Label the tick marks: $x_i = a + i\Delta x$
2. Use an approximation method:

$$L_n = \Delta x[f(x_0) + f(x_1) + \cdots + f(x_{n-1})] \quad \text{(Left endpoint)}$$
$$R_n = \Delta x[f(x_1) + f(x_2) + \cdots + f(x_n)] \quad \text{(Right endpoint)}$$
$$M_n = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)] \quad \text{(Midpoint)}$$

New - Trapezoid Rule: (all the "middle terms" are multiplied by 2)

$$T_n = \frac{1}{2}\Delta x[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

New - Simpson's Rule: *n* must be even! (Alternating multiplying middle terms by 4 and 2)

$$S_n = \frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Example: Using n = 4 subdivisions, estimate

$$\int_0^4 \sqrt{100 - x^3}dx$$

- ***Step 1***: $\Delta x = \frac{4-0}{4} = 1.$ $\quad x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$
- ***Step 2***: Here is each method:

$$(1)\left[\sqrt{100-(0)^3} + \sqrt{100-(1)^3} + \sqrt{100-(2)^3} + \sqrt{100-(3)^3}\right] \approx 38.0855 = L_4$$

$$(1)\left[\sqrt{100-(1)^3} + \sqrt{100-(2)^3} + \sqrt{100-(3)^3} + \sqrt{100-(4)^3}\right] \approx 34.0855 = R_4$$

$$(1)\left[\sqrt{100-(0.5)^3} + \sqrt{100-(1.5)^3} + \sqrt{100-(2.5)^3} + \sqrt{100-(3.5)^3}\right] \approx 36.5672$$

NEW – Trapezoid rule

$$\frac{1}{2}(1)\left[\sqrt{100-(0)^3} + 2\sqrt{100-(1)^3} + 2\sqrt{100-(2)^3} + 2\sqrt{100-(3)^3} + \sqrt{100-(4)^3}\right]$$

$$T_4 \approx 36.0855$$

NEW – Simpson's rule (n = 4 is even ✓

$$\frac{1}{3}\cdot(1)\left[\sqrt{100-(0)^3} + 4\sqrt{100-(1)^3} + 2\sqrt{100-(2)^3} + 4\sqrt{100-(3)^3} + \sqrt{100-(3)^3}\right]$$

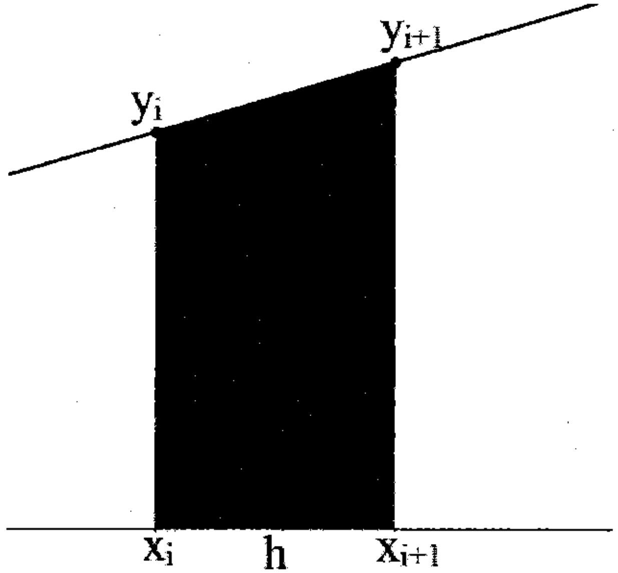
$$S_4 \approx 36.3863$$

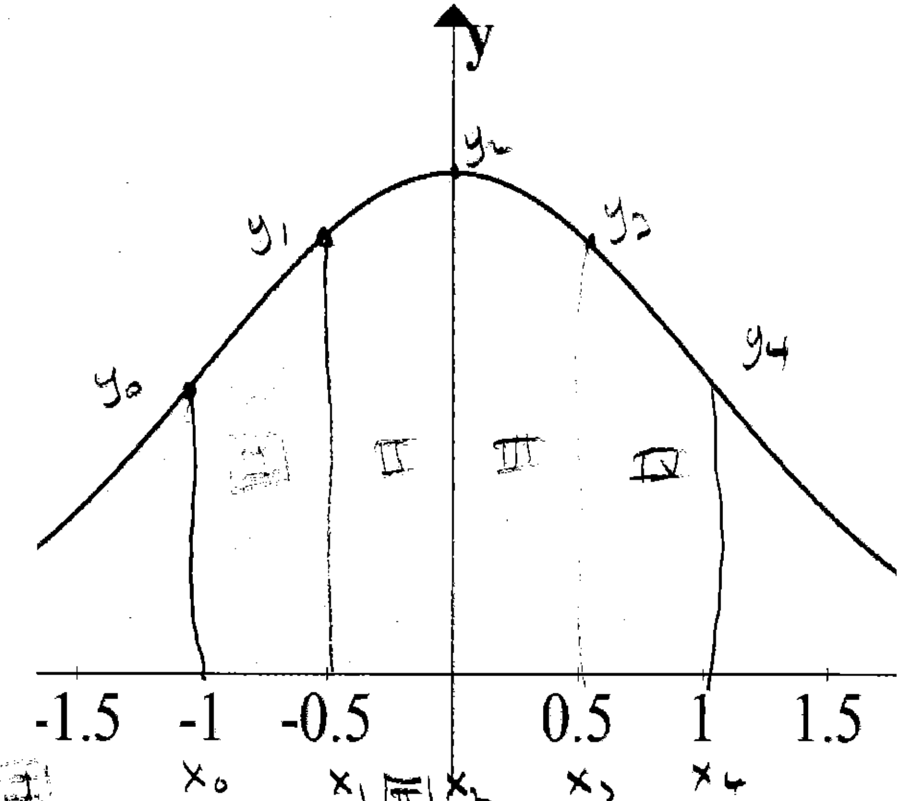
"Actual" Value (to 8 places after the decimal) ≈ 36.40897795

7.7 Derivation Notes

Trapezoid Rule:

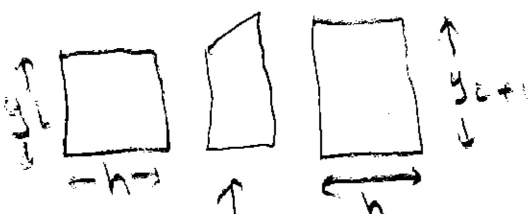
$$\text{Shaded Area } = \frac{h}{2}(y_i + y_{i+1})$$





$\frac{1}{2}\Delta x(y_0 + y_1) + \frac{1}{2}\Delta x(y_1 + y_2) + \frac{1}{2}\Delta x(y_2 + y_3) + \frac{1}{2}\Delta x(y_3 + y_4)$

$\frac{1}{2}\Delta x[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$



AREA $= hy_i$

AREA $= hy_{i+1}$

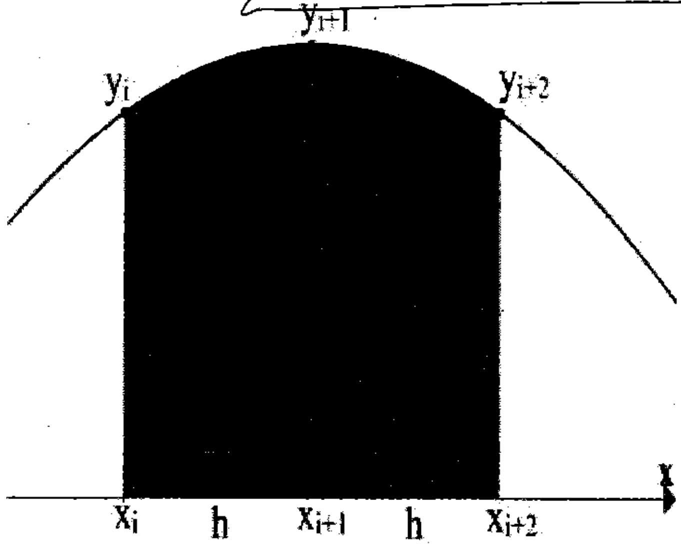
HALF-WAY BETWEEN

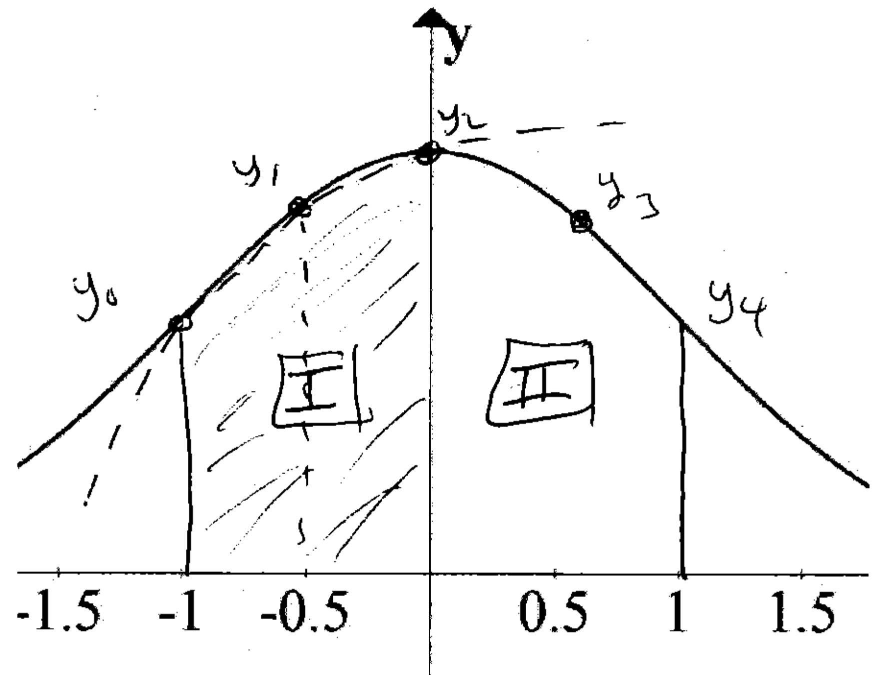
$\frac{1}{2}(hy_i + hy_{i+1}) = \frac{1}{2}h(y_i + y_{i+1})$

Simpson's Rule:

If the curve below is a ***parabola***, $y = ax^2 + bx + c$, that goes through the three indicated points, then

$$\text{Shaded Area} = \boxed{\frac{h}{3}(y_i + 4y_{i+1} + y_{i+2})}$$





$$\frac{1}{3}\Delta x(y_0 + 4y_1 + y_2) + \frac{1}{3}\Delta x(y_2 + 4y_3 + y_4)$$

$$\int_{x_i}^{x_{i+2}} ax^2 + bx + c\,dx = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\Big|_{x_i}^{x_{i+2}}$$

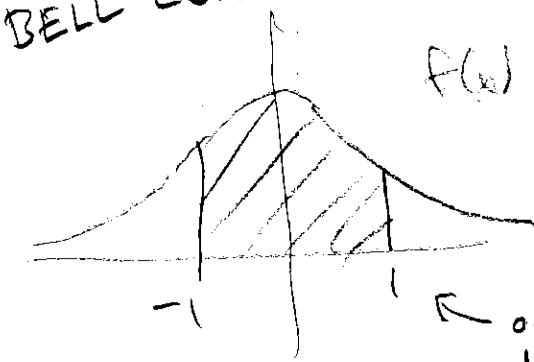
$$\frac{1}{3}\Delta x(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$

Example:
With n = 4, use both new methods to approximate (just set up)

$$\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-\frac{1}{2}x^2}\,dx$$

BELL CURVE!

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$



one standard deviation

$\Delta x = \frac{1-(-1)}{4} = \frac{1}{2}$, $x_0 = -1$, $x_1 = -\frac{1}{2}$,
$x_2 = 0$, $x_3 = \frac{1}{2}$, $x_4 = 1$

$$\frac{1}{2}\Delta x[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$$= \frac{1}{\sqrt{2\pi}}\ \frac{1}{2}\left(\frac{1}{2}\right)\left[e^{-\frac{1}{2}(-1)^2} + 2e^{-\frac{1}{2}(-\frac{1}{2})^2} + 2e^{-\frac{1}{2}(0)^2} + 2e^{-\frac{1}{2}(\frac{1}{2})^2} + e^{-\frac{1}{2}(1)^2}\right] \approx 0.672518$$

$$\frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

$$= \frac{1}{\sqrt{2\pi}}\ \frac{1}{3}\left(\frac{1}{2}\right)\left[e^{-\frac{1}{2}(-1)^2} + 4e^{-\frac{1}{2}(-\frac{1}{2})^2} + 2e^{-\frac{1}{2}(0)^2} + 4e^{-\frac{1}{2}(\frac{1}{2})^2} + e^{-\frac{1}{2}(1)^2}\right] \approx 0.6827109757$$

"ACTUAL" VALUE = 0.682689492

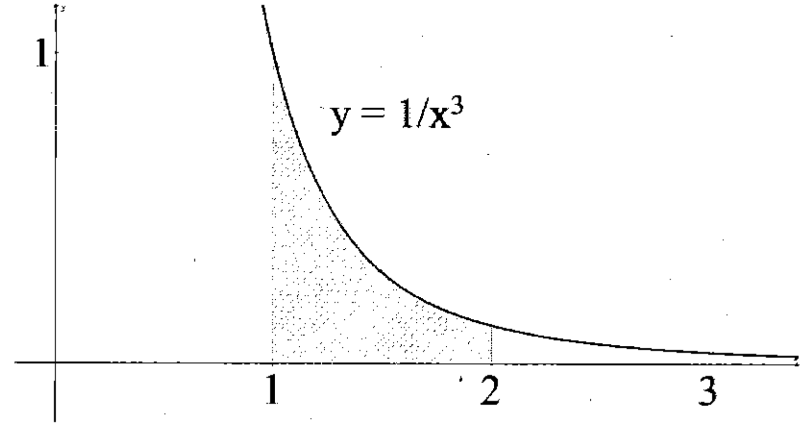
7.8 Improper Integrals *(Preview)*

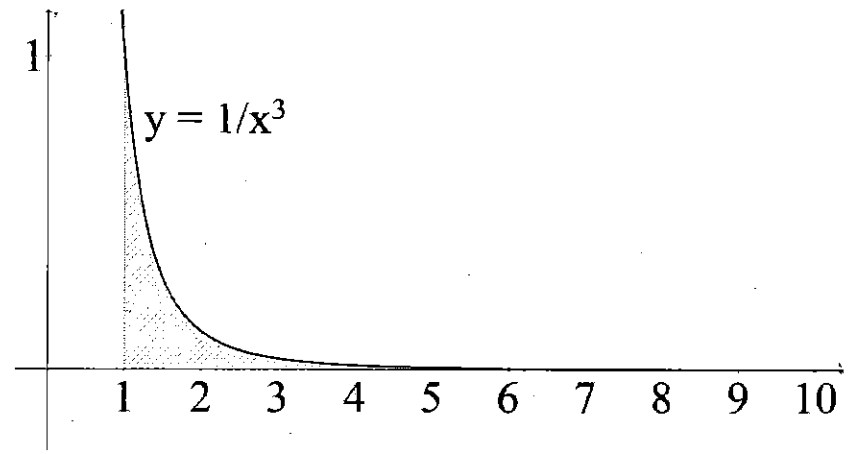
Motivation:

Consider the function $f(x) = \frac{1}{x^3}$.

Compute the area under the function…

1. …from $x = 1$ to $x = t$
2. …from $x = 1$ to $x = 10$
3. …from $x = 1$ to $x = 100$





$$\int_1^t \frac{1}{x^3}\,dx = \int_1^t x^{-3}\,dx$$

$$= \frac{1}{-2} x^{-2} \Big|_1^t$$

$$= -\frac{1}{2}\frac{1}{t^2} - -\frac{1}{2}$$

$$= \boxed{-\frac{1}{2}\frac{1}{t^2} + \frac{1}{2}}$$

So $\displaystyle\int_1^{10} \frac{1}{x^3}\,dx = -\frac{1}{2}\frac{1}{10^2} + \frac{1}{2} = 0.495$

$\displaystyle\int_1^{100} \frac{1}{x^3}\,dx = -\frac{1}{2}\frac{1}{100^2} + \frac{1}{2} = 0.49995$

Def'n: *Improper type 1 -*
infinite integral of integration

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^{t} f(x)dx$$

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_t^{b} f(x)dx$$

If the limit exists and is finite, then we say the integral *converges*.
Otherwise, we say it *diverges*.

Example:

$$\int_1^{\infty} \frac{1}{x^3}dx = \lim_{t\to\infty}\left[\int_1^t \frac{1}{x^3}\,dx\right]$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2}\frac{1}{t^2}+\frac{1}{2}\right]$$

$$= 0 + \tfrac{1}{2} = \boxed{\tfrac{1}{2}}$$

Converges!

Example:

$$\int_{-1}^{\infty} e^{-2x}\,dx = \lim_{t\to\infty} \int_{-1}^{t} e^{-2x}\,dx$$

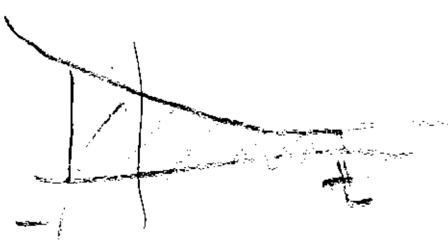
$$= \lim_{t\to\infty} \left[\frac{1}{-2} e^{-2x} \Big|_{-1}^{t} \right]$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2} e^{-2t} - -\frac{1}{2} e^{2} \right]$$

$$= \lim_{t\to\infty} \left[\underbrace{-\frac{1}{2} e^{-2t}}_{\text{SAME AS } -\frac{1}{2e^{2t}}} + \frac{1}{2} e^{2} \right]$$

$$= 0 + \frac{1}{2} e^{2} = \boxed{\frac{1}{2} e^{2}}$$

CONVERGES



Example:

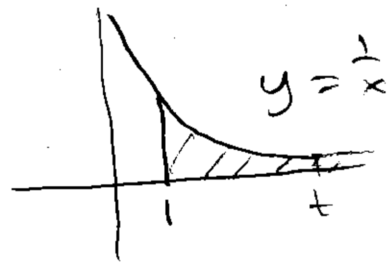
$$\int_{1}^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \left[\int_{1}^{t} \frac{1}{x}\,dx \right]$$

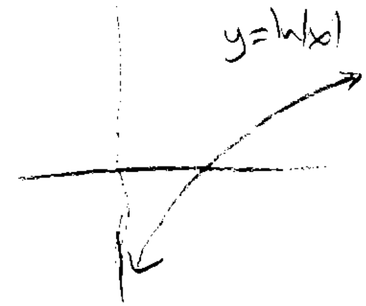
$$= \lim_{t\to\infty} \left[\ln|x| \Big|_{1}^{t} \right]$$

$$= \lim_{t\to\infty} \left[\ln(t) - \underbrace{\ln(1)}_{0} \right]$$

$$= \lim_{t\to\infty} \ln(t) = \infty$$

DIVERGES

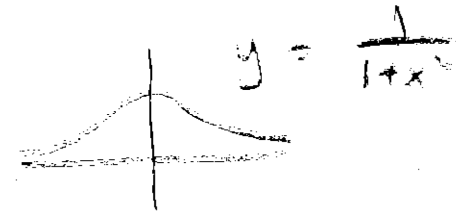




Def'n:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to-\infty}\int_{r}^{0} f(x)dx + \lim_{t\to\infty}\int_{0}^{t} f(x)dx$$

In this case, we say it *converges* only if both limits separately exist and are finite.



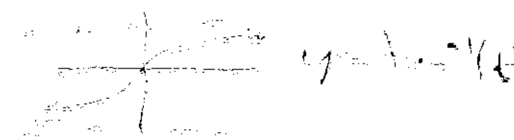
Example:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \int_{-\infty}^{0} \frac{1}{1+x^2}dx + \int_{0}^{\infty}\frac{1}{1+x^2}dx$$

$$= \lim_{r\to-\infty}\left[\int_r^0 \frac{1}{1+x^2}dx\right] + \lim_{t\to\infty}\left[\int_0^t \frac{1}{1+x^2}dx\right]$$

$$= \lim_{r\to-\infty}\left[\underbrace{\tan^{-1}(0)}_{0} - \tan^{-1}(r)\right] + \lim_{t\to\infty}\left[\tan^{-1}(t) - \underbrace{\tan^{-1}(0)}_{0}\right]$$

$$= -(-\pi/2) + \pi/2 = \boxed{\pi}$$



Def'n: *Improper type 2 - infinite discontinuity*

If $f(x)$ has a discontinuity at x = a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

If $f(x)$ has a discontinuity at x = b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

If the limit exists and is finite, then we say the integral *converges*.

Otherwise, we say it *diverges*.

Example:

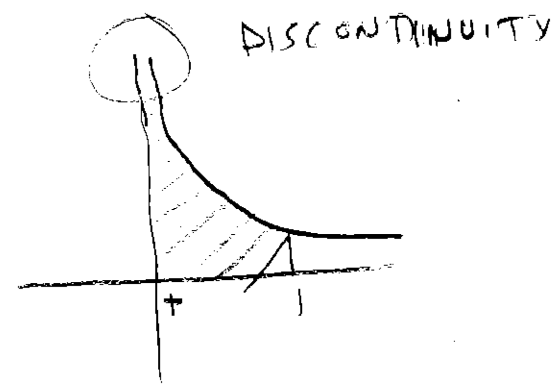
$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0^+}\left[\int_t^1 x^{-1/2}dx\right]$$

$$= \lim_{t\to 0^+}\left[2x^{1/2}\Big|_t^1\right]$$

$$= \lim_{t\to 0^+}\left[2\sqrt{1} - 2\sqrt{t}\right]$$

$$= 2 - 0 = \boxed{2}$$

CONVERGES



Example:

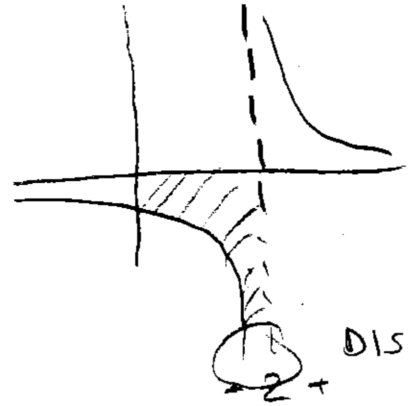
$$\int_0^2 \frac{x}{x-2}\,dx = \lim_{t\to 2^-}\left[\int_0^t \frac{x}{x-2}\,dx\right]$$

$$= \lim_{t\to 2^-}\left[\int_0^t 1 + \frac{2}{x-2}\,dx\right]$$

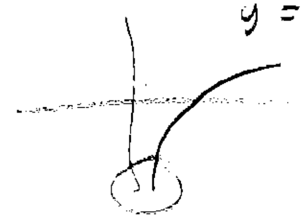
$$= \lim_{t\to 2^-}\left[\, x + 2\ln|x-2| \Big|_0^t \right]$$

$$= \lim_{t\to 2^-}\left[\left(t + \underbrace{2\ln|\underbrace{t-2}_{0}|}_{-\infty}\right) - \left(0 + 2\ln(2)\right)\right]$$

$\boxed{\text{DIVERGES}}$



$$\begin{array}{r} 1 \\ x-2\,\overline{)\,x\quad} \\ -(x-2) \\ \hline 2 \end{array}$$



If $f(x)$ has a discontinuity at $x = c$ which is **between** a and b, then

$$\int_a^b f(x)dx = \lim_{r\to c^-} \int_a^r f(x)dx + \lim_{t\to c^+} \int_t^b f(x)dx$$

In this case, we say it *converges* only if both limits separately exist and are finite.

Example:

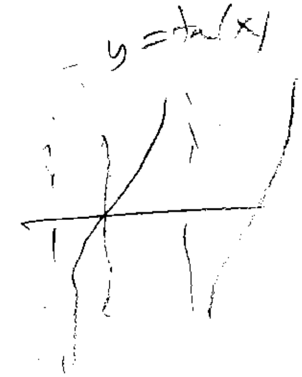
$$\int_0^\pi \frac{1}{\cos^2(x)}dx = \int_0^\pi \sec^2(x)dx$$

$\cos(x) = 0$ AT $x = \pi/2$

$$= \lim_{r\to\pi/2^-}\left[\int_0^r \sec^2(x)dx\right] + \lim_{t\to\pi/2^+}\left[\int_t^\pi \sec^2(x)dx\right]$$

$$= \lim_{r\to\pi/2^-}\left(\tan(x)\Big|_0^r\right) + \lim_{t\to\pi/2^+}\left[\tan(x)\Big|_t^\pi\right]$$

$$= \lim_{r\to\pi/2^-}\left[\underbrace{\tan(r)}_{+\infty} - 0\right] + \lim_{t\to\pi/2^+}\left[\underbrace{0 - \tan(t)}_{+\infty}\right]$$



Limits Refresher

1. If stuck, plug in values "near" t.
2. Know your basic functions/values:

$$\lim_{t\to\infty} \frac{1}{t^a} = 0, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} \frac{1}{e^{at}} = 0, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} t^a = \infty, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} \ln(t) = \infty.$$

$$\lim_{t\to 0^+} \ln(t) = -\infty.$$

3. For indeterminant forms, use algebra and/or L'Hopital's rule

Examples:

$$\lim_{t\to 1} \frac{t^2+2t-3}{t-1} \overset{\frac{0}{0}}{=} \lim_{t\to 1} \frac{(t-1)(t+3)}{(t-1)} = 4$$

$$\lim_{t\to\infty} \frac{\ln(t)}{t} \overset{\frac{\infty}{\infty}}{=} \lim_{t\to\infty} \frac{1/t}{1} = 0$$

$$\lim_{t\to\infty} t^2 e^{-3t} = \lim_{t\to\infty} \frac{t^2}{e^{3t}} \overset{\frac{\infty}{\infty}}{=} \lim_{t\to\infty} \frac{2t}{3e^{3t}} \overset{\frac{\infty}{\infty}}{=} \lim_{t\to\infty} \frac{2}{9e^{3t}} = 0$$

Aside:

A few general notes on **comparison**:

Suppose you have two functions f(x) and g(x) such that 0 ≤ g(x) ≤ f(x) for all values.

(a) If $\int_1^\infty f(x)dx$ converges, then $\int_1^\infty g(x)dx$ converges.

(b) If $\int_1^\infty g(x)dx$ diverges, then $\int_1^\infty f(x)dx$ diverges.

You can verify that

$$\int_1^\infty \frac{1}{x^p}dx\,, \qquad \text{converges for } p > 1.$$

$$\int_1^\infty e^{px}\,dx\,, \qquad \text{converges for } p < 0.$$

And you can compare off of these to sometimes quickly tell is something is converging or diverging (without calculating anything)

Ex)

$$\int_1^\infty \frac{(\sin(x)+1)}{x^2}dx$$

$$0 \le \frac{\sin(x)+1}{x^2} \le \frac{2}{x^2}$$

for all $x \ge 1$

So $\int_1^\infty \frac{\sin(x)+1}{x^2}dx \le \int_1^\infty \frac{2}{x^2}dx$

AND THIS CONVERGES!

SO THIS ALSO CONVERGES